

Unit 8 (Part A). Simultaneous-equation models

- Consider the following simple simultaneous-equation model:

$$\begin{cases} Y_t = a + \beta X_t + u_t & (1): \text{consumption function} \\ X_t = Y_t + G_t & (2): \text{income identity} \end{cases}$$

where

$$\begin{cases} Y_t = \text{consumer spending on goods and services (billions of \$)}^1 \\ X_t = \text{national income (billions of \$)} \\ G_t = \text{government spending on goods and services (billions of \$) -- the fiscal policy variable} \\ u_t = \text{the error term} \end{cases}$$

$$\begin{cases} a = \text{consumer spending irrespective income} \\ \beta = \text{marginal propensity to consume, where } 0 < \beta < 1 \end{cases}$$

¹ For simplicity, this model does not include investment spending by business firms.

$$\left\{ \begin{array}{ll} Y_t = a + \beta X_t + u_t & (1): \text{consumption function} \\ X_t = Y_t + G_t & (2): \text{income identity} \end{array} \right.$$

- Y_t and X_t are endogenous variables, since they are determined within the model.
- G_t is the fiscal policy variable and it is exogenous, since it is determined outside the model (by the government).
- Equation (1) is a **behavioral equation** (i.e., it describes the behavior of households in demanding for consumer goods and services) and requires estimation.
- Equation (2) is an **identity** (and holds for all observations $t=1, 2, \dots$) and, thus, it does not require estimation since there are no parameters in Equation 2 to be estimated.
- Equation (1) is said to be a **structural equation**, because it is behavioral and includes an endogenous variable on the right-hand side.
- a and β are the population parameters to be estimated.

- Now, we want to estimate the population parameters a and β :

$$Y_t = a + \beta X_t + u_t$$

Question: Is it appropriate to use the OLS method to estimate the population parameters a and β in the consumption function?

- **The answer is NO.** This is because Y_t and X_t in the consumption function are simultaneously determined .
For instance, X_t determines Y_t and, at the same time, Y_t determines X_t .
- In this case, the OLS estimators of a and β become **biased**.
- We refer to this bias as “**the simultaneous-equation bias**” which arises when, at least, one of the independent variables is endogenous.
- In such situations, **classical assumption 2** is violated.
- That is, for $Y_t = a + \beta X_t + u_t$ $E(X_t u_t) \neq 0$
- More specifically, due to the violation of classical assumption 2 (caused by the simultaneity), the consumption function ($Y_t = a + \beta X_t + u_t$) suffers from **a specification error problem**.

$$\left\{ \begin{array}{ll} Y_t = a + \beta X_t + u_t & (1): \text{consumption function} \\ X_t = Y_t + G_t & (2): \text{income identity} \end{array} \right.$$

- To show $E(X_t u_t) \neq 0$, we need to find the reduced-form equation for X_t by combining (1) and (2) in a way that eliminates Y_t from the right-hand side of (2). Note: A reduced-form equation includes only exogenous variables on the right-hand side.

$$\left\{ \begin{array}{ll} \bullet \text{ That is,} & X_t = (a + \beta X_t + u_t) + G_t \quad \text{or,} \quad (1 - \beta) X_t = a + G_t + u_t \\ & \text{or,} \quad X_t = a / (1 - \beta) + G_t / (1 - \beta) + u_t / (1 - \beta) \\ & \text{or,} \quad X_t = \pi_0 + \pi_1 G_t + u_t / (1 - \beta) \quad (3) \\ & \text{where} \quad \pi_0 = a / (1 - \beta) \quad \text{and} \quad \pi_1 = 1 / (1 - \beta) \end{array} \right.$$

- Equation 3 is the reduced-form equation of X_t , since the right-hand side includes only the exogenous variable G_t .
- Note: a and β are called the structural parameters, and π_0 and π_1 are called the reduced-form parameters.

$$\left\{ \begin{array}{ll} Y_t = a + \beta X_t + u_t & (1): \text{consumption function} \\ X_t = Y_t + G_t & (2): \text{income identity} \end{array} \right.$$

- Now, using the reduced-form equation of X_t , we can show that $E(X_t, u_t) \neq 0$ as follows:

$$E(X_t, u_t) = E([\pi_0 + \pi_1 G_t + u_t / (1 - \beta)] u_t) \quad \text{since } X_t = \pi_0 + \pi_1 G_t + u_t / (1 - \beta)$$

$$E(X_t, u_t) = \pi_0 E(u_t) + \pi_1 G_t \cdot E(u_t) + E([u_t / (1 - \beta)] \cdot u_t)$$

- According to classical assumption 1, $E(u_t) = 0$.

$$\text{Then, } E(X_t, u_t) = 0 + 0 + [1/(1 - \beta)] E(u_t^2) = [1/(1 - \beta)] \sigma^2 \neq 0,$$

This means that classical assumption 2 ($E(X_t, u_t) \neq 0$) is violated.

- Therefore, due to the violation of classical assumption 2 (caused by the simultaneity), the consumption function ($Y_t = a + \beta X_t + u_t$) suffers from a specification error problem.

Question: What estimation method should we use to estimate the consumption function? $Y_t = a + \beta X_t + u_t$

- Since the OLS method is inappropriate, we instead use the two-stage least squares (TSLS) estimation method.
- As the name suggests, this estimation method has two stages.

- **The first stage:** Use OLS to estimate the reduced-form function of X_t in (3): $X_t = \pi_0 + \pi_1 G_t + u_t / (1 - \beta)$
- Then, use the fitted values of X_t (denoted $FX_t = \hat{\pi}_0 + \hat{\pi}_1 G_t$) as a proxy for X_t in the second stage below.

- **The second stage:** Use OLS to estimate the following modified consumption function
$$Y_t = a + \beta FX_t + u_t \quad (4)$$
to obtain the TSLS point estimates of a and β .

- **Note 1:** X_t in the original consumption function is now replaced by FX_t (obtained from the first stage).
- **Note 2:** X_t is endogenous, but FX_t is exogenous. Why?

- Since FX_t is exogenous, for the modified consumption function ($Y_t = a + \beta FX_t + u_t$), classical assumption 2 is not violated. This is because $E(FX_t, u_t) = 0$. More specifically, $FX_t \perp u_t$.

- As such, the TSLS estimators of a and β are consistent or asymptotically unbiased.

- In addition, note that the **instrumental variables** (that we used in the first stage to obtain the fitted values of X_t) include the exogenous variable G_t plus the constant term.

- These instrumental variables (as they should) are uncorrelated with the error term u_t .

Eviews instruction:

```
LS X c G
Genr FX=X-Resid
LS Y c FX
```

- Alternatively, we can find the TSLS estimates of a and β by using the following Eview command:

```
TSLS Y c X @ c G
```

- **An important note:** The TSLS estimation method works, only when the behavioral function is identified.
- The **identification problem** will be discussed in **Part C**.